A Non-Technical Introduction to Brownian Motion

Essay 19

This essay starts a series that looks at the topic of option pricing. We start with an essay on the concept of *Brownian motion*, which has a long history in the physical sciences and a surprisingly long history in the field of finance. Readers with a strong mathematical background may be disappointed, however. Consistent with the other essays in this book, the material will be presented with a minimum of mathematics. This makes it accessible to more people, some of whom would then go on and study it in more detail. Many mathematicians and economists have consistently demonstrated to me that they cannot communicate what they are doing in simple terms, but if someone cannot explain a technical concept to a non-technical person, either the technical person does not understand the problem or it is not a very important problem.

Let us start with the assumption that the prices of assets evolve in a random manner. I do not care what the technical analysts who waste the time and money of investors say, stock prices, currency rates, and interest rates are largely unpredictable. But unpredictable does not mean one should not attempt to understand the probability process driving the numbers. Estimates of the expected returns and volatilities are essential in investing. The manner in

which asset prices fluctuate through time according to the laws of probability is known as a *stochastic process*, where the word "stochastic" means governed by the laws of probability.

A stochastic process is a sequence of observations from a probability distribution. Rolling dice at regular time intervals is a stochastic process. In this case the distribution is stable because the possible outcomes do not change from one roll to the next. Rolling a 6-5 three times in a row, while highly unlikely, in no way changes the probability of rolling another 6-5. A changing distribution, however, would be the case if we drew a card from a deck without replacing the previously drawn cards. Real world asset prices probably come from changing distributions though it is difficult to determine when the distribution has changed. Empirical analysis of past data can be useful in that context — not to predict the future but to know when the numbers are coming out according to different parameters of probability.

In around 1827, the Scottish scientist Robert Brown observed the random behavior of pollen particles suspended in water. This phenomenon came to be known as Brownian Motion. About 80 years passed before Albert Einstein, surprisingly unaware of the work of Brown, developed the mathematical properties of Brownian motion. This is not to suggest that no work was being done in the interim, but scientists did not always know what other work was being done, especially in those days. It is not surprising that it was Einstein who received most of the credit.

Let us start by assuming that a series of numbers is coming out of a standard normal (bell-shaped) probability distribution. This means that the numbers on average equal zero and have a standard deviation of 1. Just as a reminder or if you are not aware: this means that about two-thirds of the time, the numbers will be between +1 and −1, i.e., one standard deviation around the average of zero. About 95% of the time, the numbers will be between two standard deviations of the zero and 99% of the time, they will be between three standard deviations of zero.

Numbers like this have very limited properties and in this form are not very useful. Let us transform these numbers into something more useful. Suppose we are currently at time t. Take any

number you would like and call it $Z(t)$. This is our starting point. Now move ahead to time $t+1$ and draw a number from the standard normal probability distribution. Call it $e(t+1)$. A very simple transformation of the standard normal variable into the Z variable would be to add $e(t+1)$ to $Z(t)$ to obtain $Z(t+1)$. Another simple transformation would be to multiply $e(t+1)$ by a term we call t, which is the length of time that elapses between t and $t+1$. If that time interval happened to be one minute, t would be $1/[(60)(24)(365)]$, or in other words, the fraction of a year that elapses between t and $t+1$. One reason we like to multiply $e(t+1)$ by a time factor is that we would like our model to accommodate various time intervals between t and $t+1$. These statistical shocks that are the source of randomness might be larger if they were spread out over a longer time period; hence, the need to scale them by some function of time.

In fact, to model asset prices evolving continuously, we need the interval between t and $t+1$ to be as short as possible. Mathematicians say that "in the limit," meaning almost there but not quite, t will approach zero. When t is so small that it is almost but not quite zero, we use the symbol dt. Unfortunately, the model $Z(t+1) = Z(t) + e(t+1)dt$ will give us a problem when dt is nearly zero. This comes from the fact that the variance of $Z(t+1)$ will be nearly zero and zero variance is the complete absence of randomness. That is because dt is very small and to obtain the variance, we have to square it, which drives it even closer to zero. Thus, the variable Z will have no variance, which takes away its randomness. We cannot even call it a "variable" anymore. The problem is best resolved by multiplying $e(t+1)$ by the square root of dt, i.e., $Z(t+1) = Z(t) + e(t+1)(dt)^{1/2}$. Then when we need to square the expression to take the variance, we have no problems squaring $(dt)^{1/2}$, which is just dt.

This model has many convenient properties. Suppose we are interested in predicting a future value of Z, say at time s. Then the expected value of $Z(s)$ is $Z(t)$. That is because the expected random change in the process is zero. If you start off at $Z(t)$ and keep incrementing it by values that average to zero, you would not expect to get anywhere. The variance of $Z(s)$ is $d(s-t)$, or in other words, the amount of time that elapses between now, time t, and the future point, time s.

This is the process called Brownian Motion. Now let us take the difference between $Z(t+1)$ and $Z(t)$, which will be $e(t+1)(dt)^{1/2}$. We write this as $dZ(t) = e(t+1)(dt)^{1/2}$. This process, the increment to the Brownian Motion, is called a *Wiener process*, named after the American mathematician, Norbert Wiener (1894-1964), who did important work in this area. In pricing options, we are more interested in the process $dZ(t)$ than in the process $Z(t)$. We shall transform $dZ(t)$ into something more useful for modeling asset prices, but that is a topic covered in Essay 20.

It is perhaps important to note that the mathematics necessary to define the expected value and variance require the mathematical technique of integration. The ordinary rules of integration, however, do not automatically apply when the terms are stochastic. Fortunately, work by the Japanese mathematician K. Itô proved that the integral, defined as a "stochastic integral," does exist through a slightly different definition. Consequently many of the rules of ordinary integration apply.

One interesting property of the Wiener process is that when you square it, it become perfectly predictable. This seems counterintuitive. How can you generate perfectly random numbers, square them, and find them to be perfectly predictable? Well do not try any barroom bets on this one yet. No human can draw numbers fast enough to make this happen. But let us take a look at what this means. Given that $dZ(t) = e(t+1)(dt)^{1/2}$, we draw a value of $e(t+1)$. Of course it is unpredictable and we already know that its expected value is zero and its variance is the square of $(dt)^{1/2}$, that is, dt, times the variance of $e(t+1)$, which is one, so the variance is simply dt. Note that this uses the statistical rule that the variance of a constant, $(dt)^{1/2}$, times a random variable, $e(t+1)$, is the constant squared times the variance of the random variable.

Now suppose we square the value drawn. That will give us $e(t+1)^2$ times dt. Now we want to take the variance of this term. Again, to take a variance involves squaring. Once we square dt, we obtain zero, as discussed a few paragraphs back. Thus, the square of $dZ(t+1)$ has a variance of zero, which means that it is perfectly predictable and will in fact equal dt. The only problem with doing an experiment to see if you can predict it is that it relies on your ability

to make *dt* be very small. You would have to draw values of *e* faster than the speed of light. You can, however, do a reasonable replication of this phenomenon as I demonstrate in a paper I have written (citation provided below). Putting things into words, however, means that this noisy series of numbers is so small that when squaring them, we obtain an even smaller series of numbers that converges to a constant value, *dt*.

So why do these things matter? They are the foundations of the most fundamental model used to price options. We shall need to progress a little further, however, and I reserve that material for the next essay.

Brownian motion as a basis for modeling assets on which options trade was evidently discovered by the French doctoral student Louis Bachelier in 1900. Bachelier's dissertation at the Sorbonne under the direction of the famed mathematician Henri Poincaré was at that time considered to be uninteresting. It was discovered more than 50 years later by an American economist James Boness, who had it translated and reprinted. Although Bachelier solved the option pricing problem only for a very limited case, he pointed others in the right direction.

For More Reading

Bachelier, Louis. "Theory of Speculation." Reprinted in Paul H. Cootner, editor. *The Random Character of Stock Market Prices*. Cambridge: The MIT Press (1964), Ch. 2. Translated by A. James Boness.

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