

TEACHING NOTE 00-08:

CONVERGENCE OF THE BINOMIAL TO THE BLACK-SCHOLES MODEL

Version date: July 8, 2008

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It is well known that the binomial model converges to the Black-Scholes model when the number of time periods increases to infinity and the length of each time period is infinitesimally short. This proof was provided in Cox, Ross and Rubinstein (1979). Their proof, however, is unnecessarily long and relies on a specific case of the Central Limit Theorem. This is seen on pp. 250 and 252 of their article, which refers to the fact that skewness becomes zero in the limit. Also, their results are derived only for the special case where the up and down factors are given by specific formulas they obtain that allow the distribution of the stock return to have the same parameters as the desired lognormal distribution in the limit. Effectively, their distribution converges to the lognormal in the limit. They go on to show that the binomial model then converges to the Black-Scholes model under their assumptions.¹

The Cox-Ross-Rubinstein proof is elegant but far too specific. A more general proof of the convergence of the binomial to the Black-Scholes model is provided by Hsia (1983). His paper, published in *The Journal of Financial Research*, got virtually no attention; yet, it is clearly the best overall proof. It imposes no restrictions on the choice of up and down parameters. Moreover, the proof is much shorter, easier to follow, and requires few cases of taking limits.

We start with our ultimate goal, the Black-Scholes model.

$$c = SN(d_1) - Xe^{-r_c T} N(d_2)$$
$$d_1 = \frac{\log(S/X) + (r_c + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\log(S/X) + (r_c - \sigma^2/2)T}{\sigma\sqrt{T}},$$

where S is the current stock price, X is the exercise price, r_c is the continuously compounded risk-free rate, T is the time to expiration and σ^2 is the variance of the continuously compounded

¹An alternative approach is provided by Jarrow and Rudd (1983), who obtain different values for the up and down factors, which hold for any number of time steps, not just in the limit. They then provide a general sketch of how the binomial model converges to the Black-Scholes model.

return on the stock. $N(d_i)$ is the cumulative normal probability for $i = 1$ and 2 as defined above. The binomial model for the case where the option's life is divided into n time periods is

$$c = \frac{\left(\sum_{j=0}^n \binom{n}{j} p^j (1-p)^{n-j} \max(0, u^j d^{n-j} S - X) \right)}{r_0^n}$$

where $\binom{n}{j}$ is $n!/(j!(n-j)!)$ and represents the number of paths the stock can take to reach a certain point in a binomial tree, p is the risk neutral probability of an up move, u and d are one plus the return per period on the stock if it goes up and down, respectively, and r_0 is one plus the discrete interest rate per period. The numerator is the expected payout of the option at expiration under the risk neutral binomial probability, and the denominator discounts this term to the present.

This expression can be simplified. For some outcomes, $\max(0, u^j d^{n-j} S - X)$ is zero. Let a represent the minimum number of upward moves for the call to finish in the money. That is, a is the smallest integer ($a \leq n$) such that

$$u^a d^{n-a} S > X.$$

Then for all $j < a$, $\max(0, u^j d^{n-j} S - X) = 0$. For $j \geq a$, $\max(0, u^j d^{n-j} S - X) = u^j d^{n-j} S - X$. Now we need only count binomial paths from $j = a$ to n so we can write the model as

$$c = \frac{\sum_{j=a}^n \binom{n}{j} p^j (1-p)^{n-j} [u^j d^{n-j} S - X]}{r_0^n}.$$

Now let us break this up into two terms:

$$c = S \left(\frac{\sum_{j=a}^n \binom{n}{j} p^j (1-p)^{n-j} u^j d^{n-j}}{r_0^n} \right) - X r_0^{-n} \left(\sum_{j=a}^n \binom{n}{j} p^j (1-p)^{n-j} \right).$$

Let us call the two terms in the large parentheses B_1 and B_2 . The latter is the formula for the probability of a or more successes in n trials if the probability of success on any one trial is p . B_1 is

similar but cannot be expressed quite as easily without redefining the probability. Note the following:

$$\frac{p^j (1-p)^{n-j} u^j d^{n-j}}{r_0^n} = \left[\left(\frac{u}{r_0} \right) p \right]^j \left[\left(\frac{d}{r_0} \right) (1-p) \right]^{n-j}.$$

Thus, we can write this as

$$p^{*j} (1-p^*)^{n-j} \text{ where } p^* = (u/r_0)p, \text{ and } 1-p^* = (d/r_0)(1-p).$$

Thus, B_1 is a binomial probability as stated but with the probability of each trial being p^* . Now, we can write the binomial model compactly as

$$c = SB_1 - Xr_0^{-n}B_2.$$

We need this to converge to the Black-Scholes formula as given above. Obviously, we shall have to get B_1 and B_2 to converge to $N(d_1)$ and $N(d_2)$, respectively. First, however, recall that r_0^{-n} is the present value factor for n periods where the per period rate is r_0 . The per period rate can be related to an annual rate applied for T years by the relationship $r_0 = r^{1/n_a}$ where n_a is the number of periods per year. Then,

$$\begin{aligned} r_0 &= r^{1/n_a} \\ r_0^n &= r^{(1/n_a)n} \\ r_0^n &= r^T \text{ since } T = n / n_a. \end{aligned}$$

Thus, the present value factor for T years is r^{-T} . It follows that the present value of \$1 is PV where

$$\begin{aligned} PV &= r^{-T} \\ \log PV &= -T \log r \\ e^{\log PV} &= e^{-T \log r} \\ PV &= e^{-r_c T} \text{ where } r_c = \log r. \end{aligned}$$

Thus, our present value factor is equivalent to $\exp(-r_c T)$ when the interest rate is continuously compounded. So the binomial formula is equivalent to

$$c = SB_1 - X e^{-r_c T} B_2.$$

Now let us proceed to get this binomial formula to converge to the Black-Scholes formula.

Since we require $Su^a d^{n-a} > X$, it follows that

$$\begin{aligned}
& a \log u + (n - a) \log d + \log S > \log X \\
& a \log u + n \log d - a \log d + \log S > \log X \\
& a(\log u - \log d) > \log X - \log S - n \log d \\
& a > \frac{\log(X / Sd^n)}{\log(u/d)} = \frac{\log(X/S) - n \log d}{\log(u/d)}.
\end{aligned}$$

We require that a be an integer, but the formula above for a will not likely produce an integer. For example, with $S = 100$, $X = 100$, $u = 1.10$, $d = 0.95$, and $n = 10$, we have $a = 3.4988$. This means that it would take at least 4 upward moves for the option to finish in-the-money. Let us express this as follows:

$$a = \frac{\log(X/S) - n \log d}{\log(u/d)} + \zeta$$

where ζ is a number added to our computed number to make a an integer. We need to know precisely the number of upward moves to start evaluating positive payoffs of the option. In the limit ζ will converge to zero as there will be an infinite number of integer steps.

Now we proceed to work through Hsia's elegant and simple proof. We appeal to the famous DeMoivre-LaPlace limit theorem, which says that a binomial distribution converges to the normal if $np \rightarrow \infty$ as $n \rightarrow \infty$. That is, for example, for B_1 we need

$$B_1 \rightarrow \int_a^{\infty} f(j) d_j,$$

where $f(j)$ is the density for a normal distribution. Since j is not a standard normal, however, let us convert it to one by defining $z = (j - E(j))/\sigma_j$. Then we would have²

$$\int_a^{\infty} f(j) d_j = \int_d^{\infty} f(z) dz, \text{ where } d = (j - E(j))/\sigma_j.$$

Hsia, however, defines $d = -(j - E(j))/\sigma_j$. This allows him to write the above as

$$\int_{-\infty}^d f(z) dz.$$

Thus,

$$B_1 \rightarrow \int_a^{\infty} f(j) d_j = \int_{-\infty}^d f(z) dz = N(d),$$

where j has been converted to z , a standard normal. An identical procedure would be applied to B_2 .

²Do not confuse the integrating term dz with the variable d , which equals $(j - E(j))/\sigma_j$.

Now let S^* be the stock price at expiration. After n periods and j up moves, $S^*/S = u^j d^{n-j}$. Thus, the log return on the stock over the life of the option is

$$\begin{aligned}\log(S^*/S) &= j \log u + (n - j) \log d \\ &= j \log(u/d) + n \log d.\end{aligned}$$

We take the expectation,

$$E[\log(S^*/S)] = E(j) \log(u/d) + n \log d.$$

Thus,

$$E(j) = \frac{E[\log(S^*/S)] - n \log d}{\log(u/d)}.$$

The variance of the log return on the stock over the life of the option is

$$\text{Var}[\log(S^*/S)] = \text{Var}(j) [\log(u/d)]^2.$$

Thus,

$$\text{Var}(j) = \frac{\text{Var}[\log(S^*/S)]}{[\log(u/d)]^2}.$$

Since $d = (-a + E(j))/\sigma_j$, $a = (\log(X/S) - n \log d)/\log(u/d) + \zeta$, and $E(j)$ and $\text{Var}(j)$ are as given above,

$$d = \frac{\frac{\log(S/X) + E[\log(S^*/S)]}{\log(u/d)} - \zeta}{\frac{\sqrt{\text{Var}[\log(S^*/S)]}}{\log(u/d)}}.$$

From the properties of the binomial distribution, it is known that $\text{Var}(j) = nq(1 - q)$ where q is the probability per outcome.³ So

$$d = \frac{\log(S/X) + E[\log(S^*/S)]}{\sqrt{\text{Var}[\log(S^*/S)]}} - \frac{\log(u/d)}{\sqrt{q(1 - q)}} \frac{\zeta}{\sqrt{n}}.$$

As n goes to infinity, the last term clearly disappears. Our discrete binomial process is then converging to a continuous lognormal process, for which it is known that $\text{Var}[\log(S^*/S)] = \sigma^2 T$.

Thus, we have

$$d = \frac{\log(S/X) + E[\log(S^*/S)]}{\sigma \sqrt{T}}.$$

³We are not introducing a new probability q . We are just using a general specification to illustrate a point that applies to any binomial distribution. In our specific cases, q will become p and p^* .

We need this to equal d_1 and d_2 as defined by the Black-Scholes formula when the probabilities are p^* and p , respectively, as defined above. This means that we need

$$E[\log(S^*/S)] = (r_c + \sigma^2/2)T \text{ if the probability is } p^*$$

$$E[\log(S^*/S)] = (r_c - \sigma^2/2)T \text{ if the probability is } p.$$

Now let us work on the first case. First recall the value of p^* :

$$p^* = (u/r_0)p = (u/r_0)(r_0 - d)/(u - d).$$

Rearrange to solve for r_0 :

$$r_0 = [p^*(1/u) + (1 - p^*)(1/d)]^{-1}.$$

Recall that $r_0^n = r^T$ so⁴

$$r_0^n = r^T = [p^*(1/u) + (1 - p^*)(1/d)]^{-n}.$$

Now note that S/S^* can be expressed as follows:

$$S/S^* = (S_0/S_1)(S_1/S_2)\dots(S_{n-1}/S_n) = \prod_{i=1}^n (S_{i-1}/S_i),$$

where $S = S_0$ and $S^* = S_n$. The expectation of this would be

$$E(S/S^*) = \left[E \prod_{i=1}^n (S_{i-1}/S_i) \right] = \prod_{i=1}^n E(S_{i-1}/S_i).$$

Note that our ability to express the expected value of a product as the product of the expected values comes from the independence of the values. That is, in general, $\text{cov}(xy) = E(xy) - E(X)E(y)$. If x and y are independent, then $\text{cov}(xy) = 0$ and $E(xy) = E(x)E(y)$.

Now recall that our probability for B_1 is p^* . Since $S_i = S_{i-1}u$ with probability p^* and $S_i = S_{i-1}d$ with probability $1 - p^*$, then

$$E(S_{i-1}/S_i) = p^*(1/u) + (1 - p^*)(1/d).$$

Thus,

$$\begin{aligned} E(S/S^*) &= \prod_{i=1}^n [p^*(1/u) + (1 - p^*)(1/d)] \\ &= [p^*(1/u) + (1 - p^*)(1/d)]^n. \end{aligned}$$

Inverting this gives

⁴Although it is not particularly important, note that r_0 is the harmonic mean return on the stock when the probability is p^* .

$$[E(S/S^*)]^{-1} = [p^*(1/u) + (1-p^*)(1/d)]^{-n}.$$

Since $r^T = [p^*(1/u) + (1-p^*)(1/d)]^{-n}$, then $r^T = E[S/S^*]^{-1}$ or $r^{-T} = E[S/S^*]$. Therefore,

$$-T \log r = \log[E(S/S^*)].$$

So now we are working with $\log[E(S/S^*)]$. Since S^*/S is lognormally distributed, it will also be true that the inverse of a lognormal distribution is lognormally distributed.⁵ Thus, S/S^* is lognormally distributed. For any random variable x that is lognormally distributed, it will be the case that $\log[E(x)] = E[\log x] + \text{Var}[\log x]/2$. This result may not look correct, but follows from a widely used result in options: The log return is defined as $S^*/S = e^x$ so $\log(S^*/S) = x$. It is known that $E(x) = \exp[\mu + \sigma^2/2]$.⁶ Thus, $\log E[x] = \mu + \sigma^2/2$, where μ is $E[\log x]$ and σ is its standard deviation.

Since $-T \log r = \log[E(S/S^*)]$, then

$$\begin{aligned} -T \log r &= E[\log(S/S^*)] + \text{Var}[\log(S/S^*)]/2 \\ &= E[-\log(S^*/S)] + \text{Var}[-\log(S^*/S)]/2 \\ &= -E[\log(S^*/S) + \text{Var}[\log(S^*/S)]/2. \end{aligned}$$

Note where these results come from. $E[\log(S/S^*)] = E[-\log(S^*/S)] = -E[\log(S^*/S)]$, since you can always pull a minus sign out from inside an expectations operator. Also, $\text{Var}[\log(S/S^*)] = \text{Var}[-\log(S^*/S)] = \text{Var}[\log(S^*/S)]$, since you can pull the constant (-1) out in front of the variance operator by squaring it, thereby obtaining plus one times the variance.

Now what we have is

$$E[\log(S^*/S)] = T \log r + \text{Var}[\log(S^*/S)]/2.$$

Now recall that we know that $\text{Var}[\log(S^*/S)]$ is $\sigma^2 T$. Thus, we have

$$E[\log(S^*/S)] = (\log r + \sigma^2/2)T.$$

Since $\log r = r_c$, this is the result we want for B_1 to converge to $N(d_1)$.

⁵Let x be distributed lognormally meaning that $\log x$ is distributed normally. Let $y = 1/x$. Then is $\log y$ distributed normally? $\log y = \log 1 - \log x = -\log x$. Given that $\log x$ is distributed normally, changing its sign will not change its status as a normal distributed variable. It simply changes all positive outcomes to negative and vice versa. This does not change the normal distribution to some other type of distribution.

⁶This result is obtained by evaluating the integral of the normally distributed random variable x using the normal density function.

For B_2 to converge to $N(d_2)$, recall the definition $p = (r_0 - d)/(u - d)$. Then $r_0 = (pu + (1 - p)d)$. Since $S_i = S_{i-1}u$ with probability p and $S_i = S_{i-1}d$ with probability $1 - p$, then $E(S_i/S_{i-1}) = pu + (1 - p)d = r$. Since,

$$\begin{aligned} E(S^* / S) &= \left[E \prod_{i=1}^n (S_i / S_{i-1}) \right] = \prod_{i=1}^n E(S_i / S_{i-1}) \\ &= \prod_{i=1}^n [pu + (1 - p)d] = \prod_{i=1}^n [pu + (1 - p)d] \\ &= [pu + (1 - p)d]^n = r_0^n = r^T, \end{aligned}$$

it follows that

$$\log[E(S^*/S)] = T \log r.$$

Again, substituting the result that $\log[E(S^*/S)] = E[\log(S^*/S)] + \text{Var}[\log(S^*/S)]/2$, we have $E[\log(S^*/S)] = T \log r - \text{Var}[\log(S^*/S)]/2$. Recalling that $\text{Var}[\log(S^*/S)] = \sigma^2 T$, we have

$$E[\log(S^*/S)] = (\log r - \sigma^2/2)T.$$

Because $\log r = r_c$, this is the result we needed to obtain convergence of B_2 to $N(d_2)$.

Thus, the binomial model converges to the Black-Scholes.⁷ We made no assumptions regarding what the probabilities were, and we did not have to specify a particular formula for u and d . Hsia's proof requires only the condition of the DeMoivre-LaPlace limit theorem, which is that $np \rightarrow \infty$ as $n \rightarrow \infty$. This condition will not be met only if the probability per period approaches zero. This is clearly not the case. If the probability of an up move approached zero, then the probability of a down move would approach one. The model would then be meaningless as it would have no uncertainty. In fact, it is well known that the probability value converges to $1/2$.⁸

References

The key articles cited here are

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⁷Another convergence result is provided in the Rendlemen and Bartter (1979) version of the model. Their proof seems considerably more complex than even the Cox-Ross-Rubinstein proof and requires the condition that the probabilities converge to $1/2$. While this condition is met, nonetheless, it is not necessary to impose it to obtain the proof.

⁸For the Jarrow and Rudd proof, they simply set the probability to $1/2$ and then do not alter it as time steps are added (p. 188). In other applications of the binomial model, such as elsewhere in the Jarrow-Rudd book, the probability is not arbitrarily set to $1/2$, but simulations would show that it converges to $1/2$, given their formulas for u and d , as the number of time steps increases.

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