TEACHING NOTE 01-01:
ZERO COUPON BOND PRICES
AND INTEREST RATE QUOTATION CONVENTIONS

Version date: August 15, 2008 C:\Classes\Teaching Notes\TN01-01.doc

One simple but common topic of confusion in advanced as well as beginning classes is in the manner in which interest rates are quoted. It is typical to be under the misconception that a bond price is determined by applying an interest rate in some type of present value formula, when in fact a bond price is determined like any other asset: by supply and demand in the market for the asset. Given the price of a bond, the interest rate is simply an alternative way of quoting the price. An interest rate is just a non-linear transformation of the price. Even though a bond is purchased and paid for by the buyer tendering the price to the seller, it is not uncommon for the transaction to be described in terms of the interest rate. We refer to this as the interest rate "quotation."

Unfortunately, there are a number of non-linear transformations that convert a price to an interest rate. Different transformations are used in different bond markets. Why these differences exist can be answered only by reference to customs existing in those markets. This note does not provide insights into the history that has led to specific interest rate conventions; it simply explains and illustrates the differences.[1]

Let us start with some basic information. Consider the current day as time 0. We have a zero coupon bond maturing at time m and another one maturing at time n, which is later than time m. The letters m and n represent the number of days until maturity. The following bond prices are used:

B(0,m) = price of \$1 face value zero coupon bond maturing in m days

B(0,n) = price of \$1 face value zero coupon bond maturing in n days

A person who wishes to invest \$1 for n days can do so by buying an n-day bond. The value

[1]Those familiar with options should see a similarity between the interest rate on a bond and the implied volatility on an option. The bond or option price is determined in the market for the instrument. That bond or option price can then be converted to either an interest rate or a volatility, via a non-linear transformation. For options, that transformation is an option pricing model such as Black-Scholes. For bonds, the non-linear transformations are the topic of this teaching note.

$$\frac{1}{B(0,n)}$$

is the return per $1 invested after n days. In other words, this is the amount of money one would have after n days if one invested $1 in an n-day bond. Similarly,

$$\frac{1}{B(0,m)}$$

is the amount of money one would have if one invested $1 for m days.

As an alternative to investing for the longer period, n days, suppose one invested $1 for m days and simultaneously agreed with a bond dealer that when that investment matured to reinvest for n - m days. The dealer would give the investor a price quote for the purchase of another bond in m days that would mature in n days. This transaction would be a forward contract and the price would be a forward price, which we would denote as B(0,m,n). The return per $1 invested at this price would then be

$$\frac{1}{B(0,m,n)}.$$

If the investor purchased an m-day bond and entered into a forward contract to reinvest at the price B(0,m,n) for n - m days, the return per $1 invested after m days would be

$$\left(\frac{1}{B(0,m)}\right)\left(\frac{1}{B(0,m,n)}\right).$$

To avoid arbitrage, this amount should equal the return from the alternative of investing in the n-day bond.[2] That is,

$$\left(\frac{1}{B(0,m)}\right)\left(\frac{1}{B(0,m,n)}\right) = \frac{1}{B(0,n)}.$$

Solving for B(0,m,n) gives

$$B(0,m,n) = \frac{B(0,n)}{B(0,m)}.$$

It may be helpful to note that this result can also be expressed as B(0,n) = B(0,m,n)B(0,m). The significance of this format can be seen by recalling something you probably learned in basic finance. These zero coupon bond prices are just discount or

[2] If these amounts were not the same, it would be possible to purchase the n-day bond by issuing the m-day bond and going short the forward contract, or by purchasing the m-day bond and going long the forward contract, financing the position by issuing the n-day bond. If the returns from these alternative strategies are not identical, a profit can be earned with no risk and no personal funds committed.

present value factors, which we are sometimes taught to look up in tables. To find the present value of $1 for a period 0 to n, we multiply $1 by B(0,n). Alternatively, we can find the present value of $1 by multiplying $1 by the discount factor from period m to n and then by multiplying that result by the discount factor from period 0 to m. In other words, we can discount a future amount by successively discounting it back, first from n to m and then from m to 0. This is the essence of the relationship B(0,n) = B(0,m,n)B(0,m).

There are four primary interest rate quotation conventions used in financial markets. Each is, as noted earlier, just a non-linear transformation of the prices. Remember that these prices are set. Nothing we do in changing interest quotation conventions will change the prices. First, however, let us introduce the notation for the interest rates:

r(0,m) = interest rate for a bond starting today and maturing at m

r(0,n) = interest rate for a bond starting today and maturing at n (n > m)

(The above are spot rates)

r(0,m,n) = interest rate for a bond starting at m and maturing at n

(The above is a forward rate.)

These are all annual rates; thus, to apply them for periods of other than a year, we will need to make an adjustment, which you will see.

Let us complement our symbolic treatment with a numerical example. We are given the following information on 180 and 360 day zero coupon bonds:

m = 180, n = 360, n - m = 360 - 180 = 180

B(0,m) = 0.97, B(0,n) = 0.93

B(0,m,n) = 0.93/0.97 = 0.9588.

Now let us proceed to figure out the different ways of quoting the interest rates on these bonds.

The Discount Method

The discount method is primarily used in the U. S. Treasury bill market. The interest rate is multiplied by days/360, where days is the number of days to maturity, and subtracted from the assumed face value of $1. For m- and n-day bonds, we obtain the following prices:

$$B(0,m) = 1 - r(0,m)(m/360)$$

$$B(0,n) = 1 - r(0,n)(n/360)$$

$$B(0,m,n) = 1 - r(0,m,n)((n-m)/360).$$

Given the prices, the rates are found as

$$r(0,m) = (360/m)(1 - B(0,m))$$

$$r(0,n) = (360/n)(1 - B(0,n)).$$

The forward rate is just a little more complicated:

$$r(0,m,n) = \left(\frac{360}{n-m}\right)\left(1 - \frac{1 - r(0,n)(n/360)}{1 - r(0,m)(m/360)}\right).$$

Be aware that the use of 360 days is just the common convention in the T-bill market, and there is no reason why 365 cannot be used. Do note, however, that changing to 365 would not change any bond prices. It would just change the interest rate quoted.

In our example,

$$r(0,m) = (360/180)(1 - 0.97) = 0.06$$

$$r(0,n) = (360/180)(1 - 0.93) = 0.07$$

$$r(0,m,n) = \left(\frac{360}{180}\right)\left(1 - \frac{1 - 0.07(360/360)}{1 - 0.06(180/360)}\right) = 0.0825.$$

The Compound Method

The compound method makes the assumption that interest grows on itself a certain number of times a year. Once a year is common but other compounding periods are possible, including an infinite number of times a year, which is the fourth case we take up. We also need to make an assumption of how many days are in a year. We shall follow the common convention and use 365, but do recognize that 360 can be used, though as noted above, that would not change the bond price.

The bond prices are given as follows:

$$B(0,m) = \left(\frac{1}{1 + r(0,m)}\right)^{m/365}$$

$$B(0,n) = \left(\frac{1}{1 + r(0,n)}\right)^{n/365}$$

$$B(0,m,n) = \left(\frac{1}{1 + r(0,m,n)}\right)^{(n-m)/365}.$$

Given the bond prices, the rates are found as

$$r(0,m)=\left(\frac{1}{B(0,m)}\right)^{365/m}-1$$

$$r(0,n)=\left(\frac{1}{B(0,n)}\right)^{365/n}-1$$

$$r(0,m,n)=\left(\frac{(1+r(0,n))^{n/365}}{(1+r(0,m))^{m/365}}\right)^{365/(n-m)}-1.$$

In our example, the rates are

$$r(0,m)=\left(\frac{1}{0.97}\right)^{365/180}-1=0.0637$$

$$r(0,n)=\left(\frac{1}{0.93}\right)^{365/360}-1=0.0764$$

$$r(0,m,n)=\left(\frac{(1.0764)^{360/365}}{(1.0637)^{180/365}}\right)^{365/(n-m)}-1=0.0893.$$

The Add-on Method

The add-on method of quoting the interest rate is used in the Eurodollar (LIBOR) market. Given a rate, the amount invested grows by the factor 1 + rate(days/360). Again, 360 is the common convention, but other periods could be used, which would not change the bond price. It would just change the rate. The bond prices are given as follows:

$$B(0,m)=\frac{1}{1+r(0,m)(m/360)}$$

$$B(0,n)=\frac{1}{1+r(0,n)(n/360)}$$

$$B(0,m,n)=\frac{1}{1+r(0,m,n)((n-m)/360)}.$$

Given the bond prices, the rates are found as

$$r(0,m)=\left(\frac{1}{B(0,m)}-1\right)\left(\frac{360}{m}\right)$$

$$r(0,n)=\left(\frac{1}{B(0,n)}-1\right)\left(\frac{360}{n}\right)$$

$$r(0,m,n)=\left(\frac{1+r(0,n)(n/360)}{1+r(0,m)(m/360)}-1\right)\left(\frac{360}{n-m}\right).$$

In our example, the interest rates are

$$r(0,m) = \left(\frac{1}{0.97} - 1\right)\left(\frac{360}{180}\right) = 0.0619$$

$$r(0,n) = \left(\frac{1}{0.93} - 1\right)\left(\frac{360}{360}\right) = 0.0753$$

$$r(0,m,n) = \left(\frac{1 + 0.0753(360/360)}{1 + 0.0619(180/360)} - 1\right)\left(\frac{360}{180}\right) = 0.0860.$$

The Continuous Compounding Method

Annual compounding assumes that interest is paid on interest only once a year. It is obviously possible to compound more often than once a year. In fact semiannual, quarterly, and daily compounding are commonly used on such instruments as bank deposits, and we could well have included these methods here as other ways in which interest is quoted. These alternative compounding periods are, however, just transformations of the annual compounding concept. For example, $1 invested for one year at a 6% annually compounded rate grows to $1.06. If compounding were semiannual at an annual rate of 6%, the semiannual rate would be 3%. Then after one year, the $1 will have grown to $1(1.03)^2 = $1.0609. Suppose, however, that the annual rate were expressed as an equivalent semiannually compounded rate of (1.06)^{½} - 1 = .0296. Then $1 invested for two six-month periods at 2.96% would grow to (1.0296)^2 = 1.06. Thus, 2.96% is the semiannual equivalent of a 6% annual rate.

For continuous compounding, $1 invested at a rate of r grows to a value of $1exp(r(days/365)). Again, 360 days could be used, but we shall use 365. The bond prices are, therefore, as follows:

B(0,m) = exp(-r(0,m)m/365)

B(0,n) = exp(-r(0,n)n/365)

B(0,m,n) = exp(-r(0,m,n)(n - m)/365).

Given the bond prices, the rates are found as

$$r(0,m) = -\frac{\ln B(0,m)}{m/365}$$

$$r(0,n) = -\frac{\ln B(0,n)}{n/365}$$

$$r(0,m,n) = r(0,n)\left(\frac{n}{n-m}\right) - r(0,m)\left(\frac{m}{n-m}\right).$$

An interesting result that can be obtained from the forward rate equation above is that

$$r(0,n)\left(\frac{n}{365}\right) = r(0,m)\left(\frac{m}{365}\right) + r(0,m,n)\left(\frac{n-m}{365}\right).$$

which says that the long term rate is a day-weighted average of the two shorter-term rates, the m day spot rate and the n - m day forward rate.

In our example,

$$r(0,m) = -\frac{\ln 0.97}{180/365} = 0.0618$$

$$r(0,n) = -\frac{\ln 0.93}{360/365} = 0.0736$$

$$r(0,m,n) = 0.0736\left(\frac{360}{180}\right) - 0.0618\left(\frac{180}{180}\right) = 0.0854.$$

So we see that these three bonds, two spot bonds and one forward bond, can have their prices rearranged a number of different ways, the choice of which depends on the convention used in the given market. Regardless of the convention used, however, the bond price remains the same and is determined by supply and demand forces in the market.

References

Jarrow, R. and S. Turnbull. *Derivative Securities*, 2nd. ed. (Cincinnati: South-Western College Publishing). Chapter 1 of this book has a good treatment of this topic, in preparation for the advanced treatment of option pricing theory. Jarrow and Turnbull refer to these methods as “discount rates,” “discretely compounded rates,” simple interest rates,” and “continuously compounded rates.”