# TEACHING NOTE 99-05:
# RATIONAL RULES AND BOUNDARY CONDITIONS FOR OPTION PRICING

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Before one can begin to examine the models for option pricing, it is necessary to understand some fundamental principles that govern the prices of options. These principles do not give specific option prices except in a very limited sense. Rather, they define the bounds within which option prices must lie. In addition they define relationships among different options, such as those differing by exercise price, between puts and calls, and between European and American options.

We let $S_t$ be the asset price today, $S_T$ be the asset price at expiration, X be the exercise price, r be the continuously compounded risk-free rate, $\sigma$ be the volatility, and $\tau = T - t$ be the time to expiration. Let $c_t$ be the price of a European call at time t and $p_t$ be the price of a European put at time t. The asset is assumed to make no payments such as dividends, but we shall relax that assumption at appropriate points. Let $C_t$ be the price of an American call at time t and $P_t$ be the price of an American put at time t. Other notation will be introduced as needed.

**Absolute Minimum and Maximum Values**

By *absolute* minimum and maximum values, we wish to define bounds within which the option prices must lie. We do not rule out the possibility that the actual option prices might have a higher minimum or lower maximum value that we can establish later on.

The minimum value of a call is given as

$$c_t \leq S_t \quad \text{and} \quad C_t \leq S_t$$
$$p_t \leq Xe^{-r\tau} \quad \text{and} \quad P_t \leq X_t.$$

Both the European call and the American call cannot cost more than the value of the underlying asset. Either option allows the holder the right to buy the asset so the holder of the option would not pay more than the cost of the asset to acquire the right to buy the asset. A weaker statement might place this upper bound at infinity, as that is the upper bound on the asset price, but there is no reason to impose such an extreme upper bound as the current value of the asset is more precise.

A put reaches its maximum value when the asset is worthless. A European put then is worth the present value of the exercise price as its holder has the right to exercise the option at expiration and claim X dollars at that time. Thus, its current worth is the present value of X.[1] For an American put, however, there is no reason to wait as it can be immediately exercised for a cash flow of X.[2]

**The Value of an American Option Relative to the Value of a European Option**

Since an American option permits the holder to exercise the option at expiration as well as any time prior to expiration, its value must be at least as great as that of the corresponding European option:

$$C_t \geq c_t \quad \text{and} \quad P_t \geq p_t.$$

**The Value of an Option at Expiration**

At expiration both a European and an American call are worth the *intrinsic value*, which is sometimes called the *exercise value*:

$$c_T = C_T = \text{Max}(0, S_T - X).$$

If $S_T > X$ and the call is selling for less than $S_T - X$, it can be purchased and exercised, resulting in an immediate gain to the holder of $S_T - X$ less the price of the call. The ability to earn this risk-free profit will induce trading of this sort that will result in the call price increasing in value until it is equal to $S_T - X$. If $S_T \leq X$, the option should not be exercised and, hence, it expires with no value.

At expiration both a European and an American put are worth the *intrinsic value* or *exercise value*:

$$p_T = P_T = \text{Max}(0, X - S_T).$$

If $S_T < X$ and the put is selling for less than $X - S_T$, it can be purchased and exercised resulting in an immediate gain to the holder of $X - S_T$ less the price of the put.[3] The ability to earn this risk-free profit will induce trading of this sort that will result in the put price increasing in value until

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[1] It is important to understand that once an asset reaches zero, there is no recovery. Its price will be permanently fixed at zero. If there were any chance of the firm recovering, investors would gladly pay $0 for the worthless shares, which would in the worst case be worth nothing later, and in the best case, be worth something later.

[2] Though it is not our intention here to make that proof, what we have just done is formally demonstrate that there is at least one condition under which an American put will be exercised early.

[3] The put holder can either purchase the asset and use the put to sell the asset, or he can borrow the asset and sell it short by exercising the put.

it is equal to $X - S_T$. If $S_T \geq X$, the option should not be exercised and, hence, it expires with no value.

**The Lower Bound of European and American Options and The Optimality of Early Exercise**

We previously identified zero as the minimum values of European and American options. It is possible, however, to establish higher minima.

For an American call, the lower bound can be initially stated as the option's *intrinsic value*:

$$C_t \geq \text{Max}(0, S_t - X)$$

If $S_t > X$ and the call is selling for less than $S_t - X$, it can be purchased and exercised resulting in an immediate gain to the holder of $S_t - X$ less the price of the call. The ability to earn this risk-free profit will induce trading of this sort that will result in the call price increasing in value until it is worth at least $S_t - X$. In the other case, $S_t \leq X$, the option should not be exercised and, hence, we can say only that its minimum value is zero. When $S_t > X$, we say that the call is *in-the-money* and when $S_t < X$, we say that the call is *out-of-the-money*.

For a European call such a statement is not possible since it cannot be exercised immediately. It is possible, however, to make a stronger statement, via simple arbitrage arguments. Suppose we construct two portfolios, A and B, with portfolio A consisting of a long position in a European call and a risk-free pure discount bond with face value equal to the exercise price and current value equal to the present value of the exercise price. Portfolio B consists of a unit of the asset. The table below shows the structure of these portfolios, their current values and their values at expiration.

**Establishing a Lower Bound for a European Call**

| | | | Value at Expiration | |
|---|---|---|---|---|
| | **Instrument** | **Current Value** | $S_T \leq X$ | $S_T > X$ |
| **Portfolio A** | European call | $c_t$ | 0 | $S_T - X$ |
| | Bond | $Xe^{-r\tau}$ | X | X |
| | Total | $c_t + Xe^{-r\tau}$ | X | $S_T$ |
| | | | | |
| **Portfolio B** | Asset | S | $S_T$ | $S_T$ |

Portfolio A performs as well as B when $S_T > X$ and performs better when $S_T \leq X$. Portfolio A is said to *dominate* B. Consequently, the current value of A must be at least as great as the current value of B, which can be stated as $c_t + Xe^{-r\tau} \geq S$. Rearranging, we can state this as $c_t \geq S - Xe^{-r\tau}$. For the case where $S < Xe^{-r\tau}$, however, it makes little sense to state that a call price must exceed some negative value for we already know that its absolute minimum is zero. Consequently, we state formally that

$$c_t \geq \text{Max}\left(0, S_t - Xe^{-r\tau}\right).$$

For an American call, we previously noted that $C_t \geq \text{Max}(0, S_t - X)$. It is obvious, however, that $S_t - Xe^{-r\tau}$ is greater than $S_t - X$ except at the expiration ($\tau = 0$). Combined with the fact that the American call price is at least as great as the European call price, we can then state that the lower bound for the European call must also hold for the American call:

$$C_t \geq \text{Max}\left(0, S_t - Xe^{-r\tau}\right)$$

It should now be apparent that the American call will never be exercised early because its minimum value of $S_t - Xe^{-r\tau}$ is more than its value if exercised, $S_t - X$. In other words, the American call is worth more by simply selling it in the market. This may seem somewhat counterintuitive when one considers that a deep-in-the-money call might seem worth exercising. A holder of such a call might be unlikely to expect further gains but one must consider that exercise of such a call would simply result in the holder possessing the asset; If the asset is indeed going no further, it would satisfy the holder no more and, moreover, he would be out the exercise price and the interest he could continue to earn on it if he waited until expiration. It should also be apparent that exercise of a call early is equivalent to simply paying someone for an asset before it is necessary and then forgoing the right to change one's mind about its purchase at a later date. We shall soon see, however, that if the asset makes cash payments then it may be worth exercising early.

For the case where the asset makes no cash payments, the absence of early exercise will render the American and European calls equivalent in value:

$$c_t = C_t.$$

Now let us assume that the asset makes cash payments, such as dividends on a stock, over the life of the option that have a present value of D.[4] To derive the lower bound, it is necessary that we change the current value of the bond from $Xe^{-r\tau}$ to $Xe^{-r\tau} + D$. The bonds will, thus, have a value of $X + De^{r\tau}$ at expiration in either case. Portfolio B, the asset, will have a value of $S_T + De^{r\tau}$ in either case at expiration, reflecting the accumulation and reinvestment of the dividends. It should be apparent that Portfolio A is still dominant but the slight change in the composition of A leaves us with the following lower bound:

$$c_t \geq \text{Max}\left(0, S_t - D - Xe^{-r\tau}\right).$$

Unlike the case of no dividends, we now see that the lower bound of a European call can be less than $S_t$ - X, the exercise value of the American call. This would occur if $X(1 - e^{-r\tau}) < D$. In such a case, we cannot state an equivalence of the European and American call prices. In the case where the inequality is reversed, however, we have a sufficient condition for no premature early exercise of the American call and such a call would be priced as a European call.

The American call clearly has a positive probability of early exercise except in the special case of sufficiently small dividends relative to the present value of the exercise price. This establishes the fact that it may be optimal to exercise an American call early in order to capture a dividend. When the call is exercised early, the holder throws away the time value and claims the exercise value. To avoid throwing away any more time value than necessary, it is always optimal to exercise only at the last instant before the asset goes ex-dividend.

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[4]For example, if there are two dividends, $d_1$ and $d_2$ paid during the option's life and the times to the respective ex-dividend dates are $\tau_1$ and $\tau_2$, then $D = d_1\exp(-r\tau_1) + d_2\exp(-r\tau_2)$. If we can assume that the dividends are paid continuously at a rate $\delta$, then $S_te^{-\delta\tau}$ is the asset price minus the present value of the dividends; hence $D = S - (1 - e^{-\delta\tau})$.

Now we consider the lower bound of a European put. First we consider the case of no dividends. We construct Portfolio C, consisting of a European put and a unit of the asset, and Portfolio D, consisting of a risk-free bond with face value X and current value of $Xe^{-r\tau}$. The table below shows the outcomes.

**Establishing a Lower Bound for a European Put**

| | | | **Value at Expiration** | |
|---|---|---|---|---|
| | **Instrument** | **Current Value** | **$S_T \leq X$** | **$S_T > X$** |
| **Portfolio C** | European put | $P_t$ | $X- S_T$ | |
| | Asset | $S_t$ | $S_T$ | $S_T$ |
| | Total | $p_t + S_t$ | X | $S_T$ |
| | | | | |
| **Portfolio D** | Bond | $Xe^{-r\tau}$ | X | X |

Portfolio C clearly dominates Portfolio D, matching its outcome in one case and beating it in the other. Thus, the current value of C must be no less than the current value of D, giving us $p_t + S_t \geq Xe^{-r\tau}$. Rewriting this to isolate the put and noting that a negative lower bound is dominated by a lower bound of zero gives

$$p_t \geq \text{Max}\left(0, Xe^{-r\tau} - S_t\right).$$

For an American put, however, the lower bound is still $\text{Max}(0,X - S_t)$ since this exceeds the European lower bound. Consequently,

$$p_t \geq \text{Max}\left(0, X - S_t\right).$$

An American put where $S_t < X$ is said to be *in-the-money* and an American put where $S_t > X$ is said to be *out-of-the-money*.

The lower bound dominance in the case of an American call provides a simple condition under which we demonstrated that the American call will not be exercised early except in the event of a dividend. For a European put, it is sufficient to demonstrate that the case of an asset price falling to zero will trigger early exercise. The holder of the put will gain no more by waiting since the asset can go down no further, nor can it ever attain a positive value again. It is not true, however, that the holder must wait until the asset falls to zero. The exact point of early exercise is a complex matter to determine, however, and we do not cover it here.

For the case of a European put on a dividend-paying asset, we modify portfolio D so that its bond has a face value of $Xe^{-r\tau} + D$. This makes its payoff be $X + De^{r\tau}$ in either outcome while the dividends on the asset will render its value at expiration either $X + De^{r\tau}$ or $S_T + De^{r\tau}$. Thus, Portfolio C is still dominant but the resulting boundary now becomes

$$p_t \geq \text{Max}\left(0, Xe^{-r\tau} - S_t + D\right).$$

The plus sign means that dividends have a positive effect on put options. When a firm pays a dividend, it reduces its ability to grow. This is harmful to holders of calls, who benefit only from growth in the asset, but it benefits holders of puts who gain from less growth.

If the put were American, the existence of dividends renders it possible for there to be no early exercise possibility. An American put can clearly be sold for at least its minimum European value of $Xe^{-r\tau} - S_t + D$ or exercised for $X - S_t$. If $D > X(1 - e^{-r\tau})$, then it cannot be worth more to exercise it. Clearly high dividends make a put more attractive alive than exercised. If it is, however, optimal to exercise a put early, it will be done immediately after the asset goes ex-dividend.

Both American put and call option prices prior to expiration will exceed the intrinsic values because sellers of the option will bear the risk that the options will be worth substantially more than their current intrinsic values by the time expiration has arrived. The option price, thus, is said to consist of two aforementioned components, the *intrinsic value* and the *time value*, the latter reflecting the premium that disappears as expiration nears. The full price of the option - the intrinsic value plus the time value - is the objective of developing an option pricing model, a topic we cover later.

**Differences in the Values of Options Differing by Exercise Price**

Consider two European calls differing only by exercise price. The first call has an exercise price of $X_1$ and a price of $c(X_1)_t$ and the second call has an exercise price of $X_2$ and a price of $c(X_2)_t$. Construct portfolios E, consisting of a long position in the call with exercise price $X_1$ and a short position in the call with exercise price of $X_2$, and F consisting of risk-free bonds with face value of $X_2$ - $X_1$ and current value of $(X_2 - X_1)e^{-r\tau}$. We use these portfolios to establish an upper bound for the difference in the prices of the two calls.

**Establishing an Upper Bound for the Difference in the Prices of Two European Calls Differing Only by Exercise Price**

| | | | Value at Expiration | | |
|---|---|---|---|---|---|
| | **Instrument** | **Current Value** | $S_T \leq X_1$ | $X < S_T < X_2$ | $S_T \geq X_2$ |
| **Portfolio E** | Long call with strike $X_1$ | $c(X_1)_t$ | 0 | $S_T - X_1$ | $S_T - X_1$ |
| | Short call with strike $X_2$ | $-c(X_2)_t$ | 0 | 0 | $-(S_T - X_2)$ |
| | Total | $c(X_1)_t - c(X_2)_t$ | 0 | $S_T - X_1$ | $X_2 - X_1$ |
| | | | | | |
| **Portfolio F** | Bond | $(X_2 - X_1)e^{-r\tau}$ | $X_2 - X_1$ | $X_2 - X_1$ | $X_2 - X_1$ |

The first result we notice is that the payoff to the call portfolio is non-negative, which means that the initial value of the call portfolio must be non-negative. In other words,

$$c(X_1)_t \geq c(X_2)_t.$$

This means simply that the call with the lower exercise price must sell for at least as much as the call with the higher exercise price. If the calls are American, this result still holds if we can prove that the payoff of the calls is never negative. We need not worry about the call we hold for we would never exercise it early if it were to our disadvantage. If the call we are short is exercised early, then it must be the case that $S_t > X_2$, which means that $S_t > X_1$ and we could exercise our long call early, capturing early a gain of $X_2 - X_1$, the maximum payoff at expiration in the case of a European call. Thus, early exercise makes no difference and we can state that:

$$C(X_1)_t \geq C(X_2)_t.$$

The second result we notice is that Portfolio F dominates portfolio E. Consequently, we have

$$c(X_1)_t - c(X_2)_t \leq (X_2 - X_1)e^{-r\tau}.$$

This establishes an upper bound on the spread between the call prices.

If the calls are American, then we are required to modify Portfolio F such that its current value is $X_2$ - $X_1$ and its face value is ($X_2$ - $X_1$)$e^{r\tau}$. If our short call is exercised early, we simply turn around and exercise our long call, which is even deeper in-the-money, and capture a value of $X_2$ - $X_1$. This amount is invested at the risk-free rate. Without adjusting Portfolio F, we might have Portfolio E dominating F due to the interest earned on the reinvestment of $X_2$ - $X_1$. With F worth $X_2$ - $X_1$ today, however, it will grow to a value that is at least as great as that of Portfolio E in the event of early exercise. Consequently, for American calls the rule becomes

$$C(X_1)_t - C(X_2)_t \leq (X_2 - X_1)e^{-r\tau}.$$

If there is no possibility of early exercise, as is the case when the asset makes no payments, the upper bound on the American spread comes down to the upper bound on the European spread.

Now consider two European puts differing only by exercise price. The first put has an exercise price of $X_1$ and a price of $p(X_1)_t$ and the second put has an exercise price of $X_2$ and a price of $p(X_2)_t$. Construct portfolios G, consisting of a short position in the put with exercise price $X_1$ and a long position in the put with exercise price of $X_2$, and H consisting of risk-free bonds with face value of $X_2$ - $X_1$ and current value of ($X_2$ - $X_1$)$e^{-r\tau}$. We use these portfolios to set an upper bound for the difference in the prices of the two puts.

**Establishing an Upper Bound for the Difference in the Prices of Two European Puts Differing Only by Exercise Price**

| | | | Value at Expiration | | |
|---|---|---|---|---|---|
| | **Instrument** | **Current Value** | **$S_T \leq X_1$** | **$X < S_T < X_2$** | **$S_T \geq X_2$** |
| **Portfolio G** | Short put with strike $X_1$ | $-p(X_1)_t$ | $-(X_1 - S_T)$ | 0 | 0 |
| | Long put with strike $X_2$ | $p(X_2)_t$ | $X_2 - S_T$ | $X_2 - S_T$ | 0 |
| | Total | $p(X_2)_t - p(X_1)_t$ | $X_2 - X_1$ | $X_2 - S_T$ | 0 |
| | | | | | |
| **Portfolio H** | Bond | $(X_2 - X_1)e^{-r\tau}$ | $X_2 - X_1$ | $X_2 - X_1$ | $X_2 - X_1$ |

The first result we notice is that the payoff to the put portfolio is non-negative, which means that the initial value of the put portfolio must be non-negative. In other words,

$$p(X_2)_t \geq p(X_1)_t.$$

This means simply that the put with the higher exercise price must sell for at least as much as the put with the lower exercise price. If the puts are American this result still holds if we can prove that the payoff of the put portfolio is never negative. We need not worry about the put we hold for we would never exercise it early if it were to our disadvantage. If the put we are short is exercised early, then it must be the case that $S_t < X_1$, which means that $S_t < X_2$ and we could exercise our long put early, capturing early a gain of $X_2 - X_1$, the maximum payoff at expiration in the case of a European puts. Thus, early exercise makes no difference and we can state that:

$$P(X_2)_t \geq P(X_1)_t.$$

The second result we notice is that Portfolio H dominates portfolio G. Consequently, we have

$$P(X_2)_t - P(X_1)_t \leq (X_2 - X_1)e^{-r\tau}.$$

This establishes an upper bound on the spread between the put prices.

If the puts are American, then we are required to modify Portfolio H such that its current value is $X_2 - X_1$ and its face value is $(X_2 - X_1)e^{r\tau}$. If our short put is exercised early, we simply turn around and exercise our long put, which is even deeper in-the-money, and capture a value of $X_2$ - $X_1$. This amount is invested at the risk-free rate. Without adjusting Portfolio H, we might have Portfolio G dominating H due to the interest earned on the reinvestment of $X_2$ - $X_1$. With H worth $X_2$ - $X_1$ today, however, it will grow to a value that is at least as great as that of Portfolio G in the event of early exercise. Consequently, for American puts the rule becomes

$$P(X_2)_t - P(X_1)_t \leq (X_2 - X_1).$$

There is a particularly interesting relationship between the prices of European puts and calls, both differing by exercise price only. Suppose one buys the call with exercise price $X_1$, sells the call with exercise price $X_2$, buys the put with exercise price $X_2$ and sells the put with exercise price $X_1$. This constitutes what is called a *box spread*. If $S_T > X_2$, both calls are exercised and both puts are out-of-the-money for a payoff of $S_T$ - $X_1$ - ($S_T$ - $X_2$) = $X_2$ - $X_1$. If $X_1 < S_T \leq X_2$, the long call is exercised for a value of $S_T$ - $X_1$ and the long put is exercised for a value of $X_2$ - $S_T$ for a total value of $X_2$ - $X_1$. If $S_T \leq X_1$, the long put is exercised for a value of $X_2$ - $S_T$ and the short put is exercised for a value of -($X_1$ - $S_T$) for a total of $X_2$ - $X_1$. Thus, the box spread pays off $X_2$ - $X_1$ in every state. Consequently, the value of the box spread is the present value of $X_2$ - $X_1$:

$$c(X_1)_t - c(X_2)_t + p(X_2)_t - p(X_1)_t = (X_2 - X_1)e^{-r\tau}.$$

If the options are American, the analysis is only slightly modified. If the short call is exercised, then the long call is even deeper in the money and can be exercised for a value of $X_2$ - $X_1$, which is then reinvested in cash until expiration. Thus, it will grow to $X_2$ - $X_1$ plus the interest reflecting the time between the early exercise date and the expiration. At expiration, the puts will be worth zero at worst and $X_2$ - $X_1$ at best, provided they are not exercised early. If the puts are exercised early, they will be worth $X_2$ - $X_1$, which will be reinvested until expiration. In short, the box spread will pay off $X_2$ - $X_1$ at expiration or possibly before expiration. Since the options are subject to immediate early exercise, in which case they could be worth $X_2$ - $X_1$ right now and since it is possible that at some time during the life of the options another payoff of $X_2$ - $X_1$ will be received, we can say that

$$C(X_1)_t - C(X_2)_t + P(X_2)_t - P(X_1)_t \leq 2(X_2 - X_1)$$

which could also have been obtained by combining two previous results, $C(X_1)_t$ - $C(X_2)_2 \leq (X_2$ - $X_1)$ and $P(X_1)_t$ - $P(X_2)_t \leq (X_2$ - $X_1)$ .

**The Effect of Differences in Time to Expiration**

Consider two European calls differing only by time to expiration. We let them expire at $t_1$ and $t_2$ and their times to expiration will be $\tau_1 = T - t_1$ and $\tau_2 = T - t_2$. Their respective prices will be $c(\tau_1)_t$ and $c(\tau_2)_t$ with similar notation for American calls as well as European and American puts.

For a European call it is simple to demonstrate that the longer term call must sell for at least as much as the shorter-term call, in the absence of dividends. Suppose we are at the expiration date of the shorter-term call and the asset is at $S_{t_1}$. Its value is $\text{Max}(0, S_{t_1} - X)$. The longer term option, however, has time remaining of $\tau_2 - \tau_1$ so its minimum value is $\text{Max}\left(0, S_{t_1} - Xe^{-r(\tau_1-\tau_2)}\right)$, which is at least as great as the value of the shorter-term expiring option. Consequently,

$$c(\tau_2)_t \geq c(\tau_1)_t .$$

If there are dividends, then the longer term option has a minimum value of $\text{Max}\left(0, S_{t_1} - D - Xe^{-r(\tau_1-\tau_2)}\right)$, which might make it seem as if that option has a minimum less than the exercise value of the expiring option. If that is the case, however, the longer term American option would sell for at least the intrinsic value. Consequently,

$$C(\tau_2)_t \geq C(\tau_1)_t .$$

For a European put, we obtain a somewhat counterintuitive result. First assume no dividends. Then the expiring, shorter-term option is $\text{Max}(0, X - S_{t_1})$. The second option, which still has life, is worth at least $\text{Max}\left(0, Xe^{-r(\tau_1-\tau_2)} - S_{t_1}\right)$. It may well be the case that the shorter-term option is worth more. This somewhat strange result that a longer-term European put can be worth less than a shorter-term European put arises because there are conflicting sources of value from time to expiration in an option. The longer term generally helps an option, put or call, in that it gives it greater time for a favorable asset price move to occur. The longer term also has an effect arising from the present value of the potential outlay at expiration. For a put the best outcome at expiration is to exercise it, which will result in a cash inflow from sale of the asset. A longer term reduces the present value of this inflow, rendering the put potentially less valuable. This disadvantage of longer expiration will be partially, perhaps wholly, offset by the advantage of the longer time for a favorable asset price move. Puts that are deep in-the-money will tend to have the disadvantage weigh more than the advantage because their potential for exercise is greater and their potential for gains from further asset price moves is limited. All other puts will tend to have the advantage greater than the disadvantage.[5] The result is, therefore,

$$p(\tau_2)_t \gtreqless p(\tau_1)_t .$$

If the put is American, however, there is no waiting to receive the exercise price at expiration. It can always be claimed now. Thus,

$$p(\tau_2)_t \geq p(\tau_1)_t .$$

If there are dividends on the asset and the puts are European, the expiring option is worth $\text{Max}\left(0, X - S_{t_1}\right)$. The alive option is worth at least $\text{Max}\left(0, Xe^{-r(\tau_1 - \tau_2)} - S_{t_1} + D\right)$. Again, it may be the case that the longer-term put is worth less, which would, of course, all depend on the various input values. Consequently, our above statement for European puts for the no dividend case holds as well if there are dividends. If the puts are American, the remaining option will always sell for at least its intrinsic value, which makes it worth at least as much as the expiring option. Consequently, our statement for American puts without dividends holds when dividends are introduced.

**The Convexity Rule**

It is possible to derive a relationship between the prices of three options differing by exercise price. Let their exercise prices be $X_1$, $X_2$ and $X_3$ and the call prices be $c(X_1)_t$, $c(X_2)_t$, and $c(X_3)_t$. Let us construct Portfolio I consisting of $\lambda$ units of the first call and $(1-\lambda)$ units of the third call. Portfolio J consists of one unit of the second call. $\lambda$ is defined as $(X_3 - X_2)/(X_3 - X_1)$ so that $(1 - \lambda)$ is $(X_2 - X_1)/(X_3 - X_1)$. The table below shows the outcomes.

[5] For call options, the longer term is a double advantage, giving the asset more time for a favorable move and lowering the present value of the hoped-for outlay of the exercise price at expiration.

**The Relationship between the Prices of Three European Calls Differing Only by their Exercise Prices**

| | | | Value at Expiration | | | |
|---|---|---|---|---|---|---|
| | **Instru-ment** | **Current Value** | $S_T \leq X_1$ | $X_1 < S_T \leq X_2$ | $X_2 < S_T \leq X_3$ | $S_T > X_3$ |
| **Portfolio I** | Long Calls | $\lambda c(X_1)_t$ | 0 | $\lambda(S_T - X_1)$ | $\lambda(S_T - X_1)$ | $\lambda(S_T - X_1)$ |
| | Long Calls | $(1-\lambda)c(X_3)_t$ | 0 | 0 | 0 | $(1-\lambda)(S_T - X_3)$ |
| | Total | $\lambda c(X_1)_t + (1-\lambda)c(X_3)_t$ | 0 | $\lambda(S_T - X_1)$ | $\lambda(S_T - X_1)$ | $S_T - \lambda X_1 - (1-\lambda)X_3$ |
| | | | | | | |
| **Portfolio J** | Long calls | $c(X_2)_t$ | 0 | 0 | $S_T - X_2$ | $S_T - X_2$ |

When $S_T \leq X_1$, both outcomes are equal. When $X_1 < S_T \leq X_2$, Portfolio I is better than Portfolio J because $\lambda > 0$ and $S_T > X_1$. When we have $X_2 < S_T \leq X_3$, Portfolio I is better because $X_2 > X_1$. In the last case, where $S_T > X_3$, Portfolio I is equivalent to Portfolio J. This can be found by substituting the definition of $\lambda$. Putting these results together tells us that Portfolio I dominates Portfolio J. Consequently, the current value of I must be at least as great as the current value of J.

$$\lambda c(X_1)_t + (1-\lambda)c(X_3)_t \geq c(X_2)_t .$$

This is called the *convexity rule* for it states that the option price is convex with respect to the exercise price. If there are dividends on the asset, the rule is not affected since none of the positions above will collect dividends. If the options were American, Portfolio I would still dominate Portfolio J because the payoffs above at expiration would occur early. Thus,

$$\lambda C(X_1)_t + (1-\lambda)C(X_3)_t \geq C(X_2)_t .$$

Similar arguments show that the rule holds for both European and American puts. Thus,

$$\lambda p(X_1)_t + (1-\lambda)p(X_3)_t \geq p(X_2)_t$$

and

$$\lambda P(X_1)_t + (1-\lambda)P(X_3)_t \geq P(X_2)_t .$$

**Put-Call Parity**

The relationship between the price of the put and the call is referred to as *put-call parity*. We construct two portfolios called K and L. Portfolio K consists of one unit of the asset making no cash payments and one European put while Portfolio L consists of one European call and a risk-free bond with face value of X and current value of $Xe^{-r\tau}$.[6] The table below shows the outcomes at expiration.

**Put-Call Parity for European Options on Assets Making No Cash Payments**

| | | | **Value at Expiration** | |
|---|---|---|---|---|
| | **Instrument** | **Current Value** | $S_T \leq X$ | $S_T > X$ |
| **Portfolio K** | Asset | $S_t$ | $S_T$ | $S_T$ |
| | European Put | $p_t$ | $X - S_T$ | 0 |
| | Total | $S_t + p_t$ | X | $S_T$ |
| | | | | |
| **Portfolio L** | European Call | $c_t$ | 0 | $S_T - X$ |
| | Bond | $Xe^{-r\tau}$ | X | X |
| | Total | $c_t + Xe^{-r\tau}$ | X | $S_T$ |

It is clear that Portfolios K and L produce the same results at expiration; thus, they must have the same value today. We, therefore, can state put-call parity as

$$S_t + p_t = c_t + Xe^{-r\tau} .$$

[6]Portfolio K is sometimes referred to as a *protective put* while Portfolio L is sometimes referred to as a *fiduciary call.*

The above statement indicates that a long position in the asset and the put is equivalent to a long position in the call and risk-free bonds. Simple algebra reveals that

$$p_t = c_t - S_t + Xe^{-r\tau},$$

which indicates that a put is equivalent to a long call, short stock and long bonds. Likewise, a call is indicated as

$$c_t = p_t + S_t - Xe^{-r\tau},$$

which means that a call is equivalent to a long put, long stock and short bonds. The asset itself can be decomposed as follows:

$$S_t = c_t - p_t + Xe^{-r\tau},$$

which indicates that the asset can be replicated by a long call, short put and long bonds. Finally, the risk-free asset can be expressed as

$$Xe^{-r\tau} = S_t + p_t - c_t,$$

meaning that the risk-free bond is equivalent to the asset, a long put and a short call.[7]

If there are dividends on the asset, put-call parity is easily established by simply modifying the risk-free bond in Portfolio L such that its current value is $Xe^{-r\tau} + D$. At expiration Portfolio L will pay off $X + De^{r\tau}$. Portfolio K's payoff in each state will be augmented by the reinvested value of the dividends, $De^{r\tau}$. The final results will be the same in that the payoffs of the two portfolios are still equivalent, but put-call parity is now stated as

$$S_t - D + p_t = c_t + Xe^{-r\tau}.$$

In other words, the asset price is simply reduced by the present value of the dividends.

If the options are American, put-call parity is considerably more complex. We first consider the case of no dividends. Portfolio L, consisting of a long call and bonds, is not subject to early exercise since the long call would not be exercised as there are no dividends on the asset. It is necessary, however, to adjust the initial value of the bonds to X instead of $Xe^{-r\tau}$.

[7] It can also be shown that the box spread is the difference between put call parity for options with exercise price $X_1$ and options with exercise price $X_2$.

The new outcomes of Portfolio L are as follows:

*If $S_T \leq X$*

| | |
|---|---|
| $0$ | (the call) |
| $\underline{Xe^{r\tau}}$ | (the bonds) |
| $Xe^{r\tau}$ | |

*If $S_T > X$*

| | |
|---|---|
| $S_T - X$ | (the call) |
| $\underline{Xe^{r\tau}}$ | (the bonds) |
| $S_T + X(e^{r\tau} - 1)$ | |

Portfolio K, consisting of the asset and a put, is subject to early exercise. The outcomes for each of the two possible states at expiration can be different depending on whether the put were exercised early. Let us assume that the put is exercised early at time j, at which time the asset is sold short and repurchased at expiration.[8] Now let us examine the outcomes at expiration.

*If $S_T \leq X$ and the put were exercised early*:

| | |
|---|---|
| $S_T$ | (the asset) |
| $Xe^{r(T-j)}$ | (the exercised put) |
| $\underline{-S_T}$ | (cover short sale) |
| $Xe^{r(T-j)}$ | |

*If $S_T \leq X$ and the put were not exercised early:*

| | |
|---|---|
| $S_T$ | (the asset) |
| $\underline{X - S_T}$ | (the put) |
| $X$ | |

*If $S_T > X$ and the put were exercised early*:

| | |
|---|---|
| $S_T$ | (the asset) |
| $Xe^{r(T-j)}$ | (the exercised put) |
| $\underline{-S_T}$ | (cover short sale) |
| $Xe^{r(T-j)}$ | |

*If $S_T > X$ and the put were not exercised early:*

| | |
|---|---|
| $S_T$ | (the asset) |
| $\underline{0}$ | (the put) |
| $S_T$ | |

A simple comparison reveals that Portfolio L (with X as the current value of the risk-free bonds) is still dominant. Thus, $c_t + X \geq P_t + S_t$. Since this call is technically American, even though it would not be exercised early, we can say that $C_t \geq c_t$. Consequently, we can state that

[8] It is not necessary that the put be exercised with the short sale of the asset. The asset could be sold instead.

$C_t + X \geq P_t + S_t$ or $C_t - P_t \geq S_t - X$. This gives us a lower bound on the difference between the call and put prices.

We can establish an upper bound by using European put-call parity. With the condition $p_t = c_t - S_t + Xe^{-r\tau}$ and $P_t \geq p_t$, we can say that $P_t \geq c_t - S_t + Xe^{-r\tau}$. In this case, since $C_t = c_t$, we have $P_t \geq C_t - S_t + Xe^{-r\tau}$. This can then be expressed as $C_t - P_t \leq S_t - Xe^{-r\tau}$. This establishes an upper bound. Putting these two bounds together gives:

$$S_t - X \leq C_t - P_t \leq S_t - Xe^{-r\tau},$$

which is put-call parity for American options when the asset pays no dividends. The best we can do is place bounds around the differential between the call and put prices.

If the asset pays dividends with present value D, the proof is slightly more complex. We adjust the bond in Portfolio L so that its face value is D + X. Portfolio L's payoff is as follows:

*If $S_T \leq X$*

| | |
|---|---|
| $0$ | (the call) |
| $\underline{(X + D)e^{r\tau}}$ | (the bonds) |
| $(X + D)e^{r\tau}$ | |

*If $S_T > X$*

| | |
|---|---|
| $S_T - X$ | (the call) |
| $\underline{(X + D)e^{r\tau}}$ | (the bonds) |
| $S_T + De^{r\tau} + X(e^{r\tau} - 1)$ | |

Portfolio K's outcome must account for the fact that if the asset is sold short when the put is exercised early, the short seller will be liable for the dividends. Thus it will owe $De^{r(T-j)}$ at expiration.[9] The payoffs are as follows:

*If $S_T \leq X$ and the put were exercised early*:

| | |
|---|---|
| $S_T + De^{r\tau}$ | (the asset) |
| $Xe^{r(T-j)}$ | (the exercised put) |
| $\underline{-S_T - De^{r(T-j)}}$ | (cover short sale) |
| $Xe^{r(T-j)}$ | |

*If $S_T \leq X$ and the put were not exercised early:*

| | |
|---|---|
| $S_T + De^{r\tau}$ | (the asset) |
| $\underline{X - S_T}$ | (the put) |

[9]Technically the dividends are due when they are paid by the firm but we can assume that the short seller borrows the amount of the dividends and repays them at expiration or that instead of paying the dividends, he promises to pay the dividends with accumulated interest at expiration.

| | | | |
|---|---|---|---|
| $+ D(e^{r\tau} - e^{r(T-j)})$ | | $X + De^{r\tau}$ | |
| *If $S_T > X$ and the put were exercised early*: | | *If $S_T > X$ and the put were not exercised early:* | |
| $S_T + De^{r\tau}$ | (the asset) | $S_T + De^{r\tau}$ | (the asset) |
| $Xe^{r(T-j)}$ | (the exercised put) | $\underline{0}$ | (the put) |
| $\underline{-S_T - De^{r(T-j)}}$ | (cover short sale) | | |
| $Xe^{r(T-j)}$ | | $S_T + De^{r\tau}$ | |
| $+ D(e^{r\tau} - e^{r(T-j)})$ | | | |

In each case, Portfolio L dominates Portfolio K. Consequently we can say that $c_t + X + D \geq S_t + P_t$. Rewriting, we have, $S_t + P_t \leq c_t + X + D$. With $C_t \geq c_t$, we can, therefore, say that $S_t + P_t \leq C_t + X + D$. This establishes a lower bound on $C_t - P_t$ of $S_t - D - X \leq C_t - P_t$. An upper bound can be established by noting that for a non-dividend paying asset, $C_t - P_t \leq S_t - Xe^{-r\tau}$. Since the imposition of dividends decreases a call's value and increases a put's value, this statement must also be true if we introduce dividends. Combining our two statements gives us put-call parity for American options with dividends on the underlying asset:

$$S_t - D - X \leq C_t - P_t \leq S_t - Xe^{-r\tau}.$$

**The Effect of Interest Rates on Option Prices**

Interest rates impart a small but positive effect on call option prices and a small but negative effect on put option prices.[10] Consider that the holder of a European call faces a payoff at expiration of either zero or $S_T - X$. If interest rates increase, the value of the possible zero payoff is unaffected but the present value of the X dollars paid out if the option ends up in the money is less. For the holder of a European put option, the payoff at expiration is either zero of $X - S_T$. If interest rates increase, the value of the possible zero payoff is unaffected but the present value of the receipt of X dollars is lower. Consequently, rising interest rates decrease the value of the put.

[10] It is important to remember that we are examining the effect of an interest rate change alone. If interest rates increase and this causes asset prices to decrease, as is often the case, then the option price may not change in the direction indicated in the discussion that follows. Here we are strictly holding secondary effects constant in order to see how interest rates alone affect option prices.

If the options are American, these statements are correct as long as the options should not be exercised immediately. As long as there is a possibility of a payoff at some future date, then the present value of the cash outflow at exercise of the call or cash inflow at exercise of the put is affected by the interest rate. In the trivial case that the options should be exercised immediately, their values are $S_t$ - X or X - $S_t$ and are unaffected by the interest rate.

There are a variety of other explanations for the effect of interest rates on option prices. Most rely on the idea that the call is a leveraged transaction that substitutes for a stock margin trade and that the put is like an insurance policy.[11]

**The Effect of Volatility on Option Prices**

If it is not already intuitively obvious, it is simple to demonstrate that a call option on an asset with higher volatility will be worth more, all else equal.[12] For example, consider two options expiring at the same future time. The options can be European or American. Let the exercise prices be X and the underlying asset prices both be $S_t$. The first option, however, is on an asset whose volatility is $\sigma_1$ while the second option is on an asset whose volatility is $\sigma_2$. Let us describe the volatility of the first option as represented by two possible asset prices at expiration: $S_t + \gamma_1$ and $S_t - \gamma_1$ and for the second option: $S_t + \gamma_2$ and $S_t - \gamma_2$. We let $\gamma_2 > \gamma_1$. The possible payoffs of the first option at expiration are $\text{Max}(0, S_t + \gamma_1 - X)$ and $\text{Max}(0,S_t - \gamma_1 - X)$. The possible payoffs on the second option are $\text{Max}(0, S_t + \gamma_2 - X)$ and $\text{Max}(0,S_t - \gamma_2 - X)$. Let us assume that for both options, the higher payoff is in-the-money and the lower payoff is out-of-the-money. With $\gamma_2 > \gamma_1$ as is required for $\sigma_2 > \sigma_1$, we see that the second option's payoffs are as good as those of the first option in some states and better in others. Consequently,

$$c(\sigma_2)_t \geq c(\sigma_1)_t .$$

This result will hold regardless of whether there are dividends. In addition, since an American call need not be exercise early, it can be made to produce the same payoffs as a European call; hence,

$$C(\sigma_2)_t \geq C(\sigma_1)_t .$$

Similar arguments would show that the results hold for European and American puts. Hence,

[11] Chance (1994) has explored these explanations and provided full interpretations of the effect of various inputs on option prices using the comparative statics of the Black-Scholes option pricing model.

[12] It is also important to note here as well that we are abstracting from any secondary effects of a volatility change on the price of the asset. We hold all other factors constant and examine only the effect of a volatility change.

$$p(\sigma_2)_t \geq p(\sigma_1)_t$$

and for American puts,

$$P(\sigma_2)_t \geq P(\sigma_1)_t.$$

These results also mean that if the volatility of the asset changes, the value of the option will change in the same direction.

There has been extensive empirical research on whether option prices conform to the rules given here. The overwhelming evidence has been that option prices do conform, as surely they must since arbitrage is quickly and easily exploited except for the very smallest deviations that are consumed by transaction costs. These articles are cited in the references.

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See also the following paper for an interpretation of the effect of interest rates on option prices:

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