Winning the Tower of Hanoi Game
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The Tower of Hanoi is a game in which n round disks of successively smaller sizes are placed on a vertical peg. To the right of that peg are two more pegs with no disks on them. The object is to move all n disks from the first peg to the third. You can move only one peg at a time, and you cannot move a larger peg on to a smaller peg. It is a challenging game. Perhaps you will call it the Tower of Annoy! But in fact, there is a logical progression that leads to victory. Of course, once you figure out the logic, you may never play it again.

As noted, there are n disks. I think the somewhat standard case is $n = 5$, but yu can use as few as three or as many as you like. A mathematical property of the game is that the minimum number of moves to win is $2^n - 1$. Thus, with three disks, it will take $2^3 - 1 = 7$ moves. With four disks, it will take $2^4 - 1= 15$ moves. With 5 disks $2^5 - 1 = 31$.

Rather than introduce diagrams that actually look like the game, which requires a lot of effort I'm not willing to make, I am going to use tables that look like the layout. This will serve quite adequately to explain how to win the game.

I will illustrate it with $n = 5$ disks. I will call them Disks 1, 2, 3, 4 and 5 with 5 being the largest and 1 the smallest. I'll call the pegs A, B, and C. This is how the arrangement looks at the start. Each row is a slot where a disk can go and each column is a peg.

1		
2		
3		
4		
5		
Peg A	Peg B	Peg C

This is the goal.

		1
		2
		3
		4
		5
Peg A	Peg B	Peg C

Of course, there is no reason why the winning position cannot be all disks on Peg B, but typically it is necessary to move the disks to Peg C. This is simply to mess you up. By putting Peg B between you and your goal, the game is designed to tempt you to make an intermediate stop of Disk 5 from Peg A to Peg B before moving it to Peg C that is completely unnecessary. Instead, if you follow my advice, you will move the disks with efficiency and get to your destination with no extra moves. If you have a plan, you will get there in th shortest possible time. Otherwise, you will ramble. You might finally get there, but it won't be the best route.

To win the game, you need to break it up into parts. Consider that you have to be able to move Disk 5 from Peg A to Peg C. This requires that you get to the following:

	1	
	2	
	3	
5	4	
Peg A	Peg B	Peg C

Then you can move 5 from A to C. You still have to get Disk 1, 2, 3 and 4 from B to C, but we'll cover that. Let's first see how to get the arrangement above with 1, 2, 3, and 4 to B. We have to get 4 into B and the other three into C, like this.

		1
		2
5	4	3
Peg A	Peg B	Peg C

This is our initial objective. Now, I want to introduce some rules. These are not rules of the game, but rather, rules you should remember in order to make progress in small efficient parts. These rules apply only to moving two or three disks to an open peg, or a peg that contains a disk larger than the largest one you want to move.

Rule of 2s: If you have two disks you want to move to another peg, do the following:

1. Move the first, smallest disk to the peg you don't want to move the disks to.
2. Move the second disk to the peg you want to move the disks to.
3. Move the first disk to the peg you want to move the disks to.

And you're done. It took three moves, which is consistent with the $2^n - 1$.

Rule of 3s: If you need to move three disks from one peg to another. Here are the steps.

1. Move the top, smallest disk to the peg you want to get the three disks on.
2. Move the second disk to the other peg.
3. Move the first disk on top of the second
4. Move the third disk to the disk you want the three on.
5. Move the first disk back to the original peg.
6. Move the second disk to the peg you want the three on.
7. Move the first to the peg you want the three on.

Notice that these moves are consistent with the $2^n - 1$ formula. Also, notice at various stages of the Rule of 3s, you applied the Rule of 2s.

With five disks, as noted above, it takes $2^5 - 1 = 31$ moves, provided you do it correctly. You can of course do it with more moves, but that involves backtracking and wasting moves. Here is the right way to do it.

First, use the Rule of 3s. You want to get the three smallest disks in Peg A to Peg C. This will take seven moves.

Move 1: Move Disk 1 from Peg A to Peg C

2		
3		
4		
5		1
Peg A	Peg B	Peg C

That is, we want to move Disks 1, 2, and 3 from A to B, so we had to first move 1 to C. Outside of moving disk 1 a second time in a row, which you should never do, you now have only one choice.

Move 2: Move Disk 2 from Peg A to Peg B

3		
4		
5	2	1
Peg A	Peg B	Peg C

Move 3: Move Disk 1 from Peg C to Peg B

3		
4	1	
5	2	
Peg A	Peg B	Peg C

Move 4: Move Disk 3 from Peg A to Peg C

4	1	
5	2	3
Peg A	Peg B	Peg C

Now we need to get Disk 2 from Peg B to Peg C, but Disk 1 is in the way. But that is easy to fix. We temporarily get Disk 1 out o the way. We're still inside the Rule of 3s, but we now need to apply the Rule of 2s. This will take three moves.

Move 5: Move Disk 1 from Peg B to Peg A

1		
4		
5	2	3
Peg A	Peg B	Peg C

Move 6: Move Disk 2 from Peg B to Peg C

1		
4		2
5		3
Peg A	Peg B	Peg C

Move 7: Move Disk 1 from Peg A to Peg C

		1
4		2
5		3
Peg A	Peg B	Peg C

Like I said, it will take seven moves to apply the Rule of 3s and three moves to apply the Rule of 2s. Now we're making real progress.

Move 8: Move Disk 4 from Peg A to Peg B

		1
		2
5	4	3
Peg A	Peg B	Peg C

I guess that would be a Rule of 1s, but I didn't think I had to go over that one. This achieves our initial goal of getting Disk 4 into Peg B. Next, we need to get disks 1, 2, and 3 from Peg C to Peg B. Remember the Rule of 3s. This should take 7 moves. So, after move 15, we'll be ready to move Disk 5 to Peg C.

Move 9: Move Disk 1 from Peg C to Peg B

	1	2

5	4	3
Peg A	Peg B	Peg C

Outside of moving Disk 1 again, we have only one move.

Move 10: Move Disk 2 from Peg C to Peg A

2	1	
5	4	3
Peg A	Peg B	Peg C

We have to get Disk 3 on top of disk 4, so we need to get Disk 1 out of the way.

Move 11: Move Disk 1 from Peg B to Peg A

1		
2		
5	4	3
Peg A	Peg B	Peg C

Move 12: Move Disk 3 from Peg B to Peg C

1		
2	3	
5	4	
Peg A	Peg B	Peg C

We need to get Disk 2 on top of Disk 3, but Disk 1 is in the way. Easy to fix.

Move 13: Move Disk 1 from Peg A to Peg C

2	3	
5	4	1
Peg A	Peg B	Peg C

Move 14: Move Disk 2 from Peg A to Peg B

	2	
	3	
5	4	1
Peg A	Peg B	Peg C

Hopefully the next move is obvious.

Move 15: Move Disk 1 from Peg C to Peg B

	1	
	2	
	3	
5	4	
Peg A	Peg B	Peg C

Just like we said, after 15 moves, we're now in a position to move Disk 5 to Peg C, though we still have 16 moves to go.

Move 16: Move Disk 5 from Peg A to Peg C

	1	
	2	
	3	
	4	5
Peg A	Peg B	Peg C

It should be apparent that we need to get Disks 1, 2, and 3 onto Peg A in order to clear Disk 4 so it can be moved to Peg C. Remember the Rule of 3s. Since the Rule of 3s requires seven moves, after the 23rd move, we ought to be able to move Disk 4 onto Disk 5.

Move 17: Move Disk 1 from Peg B to Peg A

	2	
	3	
1	4	5
Peg A	Peg B	Peg C

Move 18: Move Disk 2 from Peg B to Peg C

	3	2
1	4	5
Peg A	Peg B	Peg C

Move 19: Move Disk 1 from Peg A to Peg C

		1
	3	2
	4	5
Peg A	Peg B	Peg C

Move 20: Move Disk 3 from Peg B to Peg A

		1
		2
3	4	5
Peg A	Peg B	Peg C

Now, we need to get Disks 1 and 2 from Peg C to Peg A. This is the Rule of 2s.

Move 21: Move Disk 1 from Peg C to Peg B

	1	2
3	4	5
Peg A	Peg B	Peg C

Move 22: Move Disk 2 from Peg C to Peg A

2	1	
3	4	5
Peg A	Peg B	Peg C

Now we want to move Disk 4 from Peg B to Peg C, but Disk 1 is in the way.

Move 23: Move Disk 1 from Peg B to Peg A

1		
2		
3	4	5
Peg A	Peg B	Peg C

Just as we said, after the 23rd move, we can now move Disk 4 to Peg C.

Move 24: move Disk 4 from Peg B to Peg C

1		
2		4
3		5
Peg A	Peg B	Peg C

The home stretch is, again, an application of the Rule of 3s.

Move 25: Move Disk 1 from Peg A to Peg C

		1
2		4
3		5
Peg A	Peg B	Peg C

Move 26: Move Disk 2 from Peg A to Peg B

		1
		4
3	2	5
Peg A	Peg B	Peg C

Move 27: Move Disk 1 from Peg C to Peg B

	1	4
3	2	5
Peg A	Peg B	Peg C

Move 28: Move Disk 3 from Peg A to Peg C

		3
	1	4
	2	5

Peg A	Peg B	Peg C

Though still inside the Rule of 3s, we use the Rule of 2s. That means we'll be done in three moves.

Move 29: Move Disk 1 from Peg B to Peg A

		3
		4
1	2	5
Peg A	Peg B	Peg C

Move 30: Move Disk 2 from Peg B to Peg C

		2
		3
		4
1		5
Peg A	Peg B	Peg C

One more move.

Move 31: Move Disk 1 from Peg A to Peg C

		1
		2
		3
		4
		5
Peg A	Peg B	Peg C

That's it. 31 moves. Now, you can win the game with more than 31 moves, but that involves a lot of backtracking, so learn to do it with efficiency.

What about a Rule of 4s? This ought to take 15 moves. Indeed, we could have used it here, but think about it. If you need to move four disks, you can simply figure where the top three disks need to be in order to free the peg on which you will move the largest disk. So, you can simply apply the Rule of 3s twice, and you'll need only one more move.

What about playing the game with more than five pegs? Indeed, you can make the game a bit more challenging by using more pegs. With six pegs, it will take $2^6 - 1 = 63$ moves. Break the game up into parts and apply the Rules of 2s and 3s.